CHAPTER 9

COOLING AND FREEZING TIMES OF FOODS

PRESERVATION of food is one of the most significant applications of refrigeration. Cooling and freezing of food effectively reduces the activity of micro-organisms and enzymes, thus retarding deterioration. In addition, crystallization of water reduces the amount of liquid water in food items and inhibits microbial growth (Heldman 1975).

In order for cooling and freezing operations to be cost-effective, refrigeration equipment should fit the specific requirements of the particular cooling or freezing application. The design of such refrigeration equipment requires estimation of the cooling and freezing times of foods and beverages, as well as the corresponding refrigeration loads.

Numerous methods for predicting the cooling and freezing times of foods and beverages have been proposed, including those based on numerical, analytical, and empirical analysis. The designer is faced with the challenge of selecting an appropriate estimation method from the many available methods. This chapter reviews selected procedures available for estimating the cooling and freezing times of foods and beverages, and presents examples of these procedures.

THERMODYNAMICS OF COOLING AND FREEZING

The cooling and freezing of food is a complex process. Prior to freezing, sensible heat must be removed from the food to decrease its temperature from the initial temperature to the initial freezing point of the food. This initial freezing point is somewhat lower than the freezing point of pure water due to dissolved substances in the moisture within the food. At the initial freezing point, a portion of the water within the food crystallizes and the remaining solution becomes more concentrated. Thus, the freezing point of the unfrozen portion of the food is further reduced. As the temperature continues to decrease, the formation of ice crystals increases the concentration of the solutes in solution and depresses the freezing point further. Thus, the ice and water fractions in the frozen food depend on temperature.

The cooling and freezing of foods and beverages can be described via the Fourier heat conduction equation:

$$\frac{\partial T}{\partial \theta} = \frac{1}{\rho c}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] \tag{1}$$

where T is temperature, θ is time, ρ is the density of the food, c is the specific heat of the food, k is the thermal conductivity of the food and x, y, and z are the coordinate directions. For regularly shaped food items with constant thermophysical properties, uniform initial conditions, constant external conditions and a prescribed surface temperature or a convection boundary condition, exact analytical solutions for Equation (1) exist. However, for practical cooling and freezing processes, foods are generally irregularly shaped with temperature dependent thermophysical properties. Therefore, exact analytical solutions cannot be derived for the cooling and freezing times of foods and beverages.

Accurate numerical estimations of the cooling and freezing times of foods can be obtained using appropriate finite element or finite difference computer programs. However, the effort required to perform this task makes it impractical for the design engineer. In addition, two-dimensional and three-dimensional simulations require time consuming data preparation and significant computing time. Hence, most of the research effort has been in the development of semi-analytical/empirical cooling and freezing time prediction methods that make use of simplifying assumptions.

COOLING TIMES OF FOODS AND BEVERAGES

Before a food item can be frozen, the temperature of the food must be reduced to its initial freezing point. This cooling process, also known as precooling or chilling, removes only sensible heat and thus, no phase change occurs.

The cooling time of foods and beverages is influenced by the ratio of the external heat transfer resistance to the internal heat transfer resistance. This ratio, called the Biot number, is defined as

$$\mathrm{Bi} = hL/k \tag{2}$$

where h is the convective heat transfer coefficient, L is the characteristic dimension of the food item, and k is the thermal conductivity of the food item.

When the Biot number approaches zero ($\mathrm{Bi} < 0.1$), the internal resistance to heat transfer is much less than the external resistance, and the lumped-parameter approach can be used to determine the cooling time of a food item (Heldman 1975). When the Biot number is very large ($\mathrm{Bi} > 40$), the internal resistance to heat transfer is much greater than the external resistance, and the surface temperature of the food item can be assumed to be equal to the temperature of the cooling medium. For this latter situation, series solutions of the Fourier heat conduction equation are available for simple geometric shapes. When the Biot number falls within the range $0.1 < \mathrm{Bi} < 40$, both the internal resistance to heat transfer and the convective heat transfer coefficient must be considered. In this case, series solutions, which incorporate transcendental functions to account for the influence of the Biot number, are available for simple geometric shapes.

Simplified methods for predicting the cooling times of foods and beverages are applicable to regularly and irregularly shaped food items over a wide range of Biot numbers. In this chapter, these simplified cooling time estimation methods are grouped into two main categories: (1) methods based on f and j factors, and (2) methods based on equivalent heat transfer dimensionality. Furthermore, the methods based on f and j factors are divided into two subgroups: (1) methods for regular shapes, and (2) methods for irregular shapes.

The preparation of this chapter is assigned to TC 10.9, Refrigeration Application for Foods and Beverages.

Cooling Time Estimation Methods Based on *f* and *j* Factors

All cooling processes exhibit similar behavior. After an initial lag, the temperature at the thermal center of the food item decreases exponentially (Cleland 1990). As shown in Figure 1, a cooling curve depicting this behavior can be obtained by plotting, on semilogarithmic axes, the fractional unaccomplished temperature difference versus time. The fractional unaccomplished temperature difference Y is defined as follows:

$$Y = \frac{T_m - T}{T_m - T_i} = \frac{T - T_m}{T_i - T_m} \quad (3)$$

where T_m is the cooling medium temperature, T is the product temperature, and T_i is the initial temperature of the product.

This semilogarithmic temperature history curve consists of one initial curvilinear portion, followed by one or more linear portions. Empirical formulas, which model this cooling behavior, incorporate two factors, f and j, which represent the slope and intercept, respectively, of the temperature history curve. The j factor is a measure of the lag between the onset of cooling and the exponential decrease in the temperature of the food. The f factor represents the time required to obtain a 90% reduction in the non-dimensional temperature difference. Graphically, the f factor corresponds to the time required for the linear portion of the temperature history curve to pass through one log cycle. The f factor is a function of the Biot number while the j factor is a function of the Biot number and the location within the food item.

The general form of the cooling time model is

$$Y = \frac{T_m - T}{T_m - T_i} = je^{-2.303\theta/f} \quad (4)$$

where θ is the cooling time. This equation can be rearranged to give the cooling time explicitly as

$$\theta = \frac{-f}{2.303}\ln\left(\frac{Y}{j}\right) \quad (5)$$

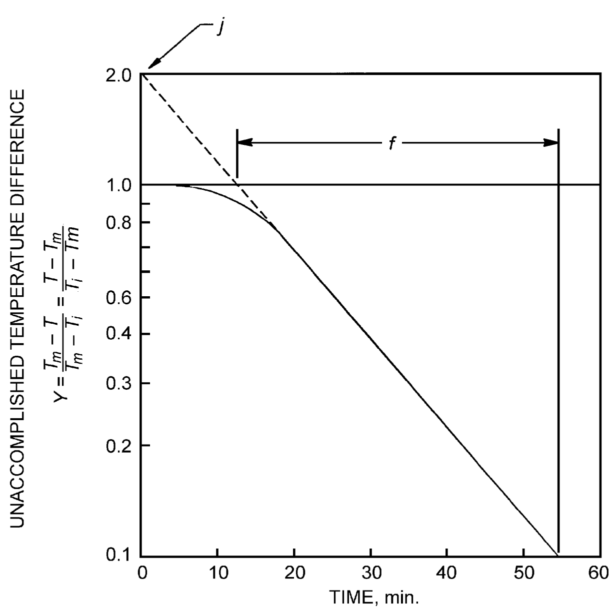

Fig. 1 Typical Cooling Curve

Determination of *f* and *j* Factors for Slabs, Cylinders, and Spheres

From analytical solutions, Pflug et al. (1965) developed charts for determining f and j factors for food items shaped either as infinite slabs, infinite cylinders, or spheres. For this development, they assumed a uniform initial temperature distribution in the food item, a constant surrounding medium temperature, convective heat exchange at the surface, and constant thermophysical properties. Figure 2 can be used to determine f values while Figures 3, 4, and 5 can be used to determine j values. Because the j factor is a function of location within the food item, Pflug et al. presented three charts for determining j factors: one corresponding to center temperature, one corresponding to volumetric average temperature and one corresponding to surface temperature.

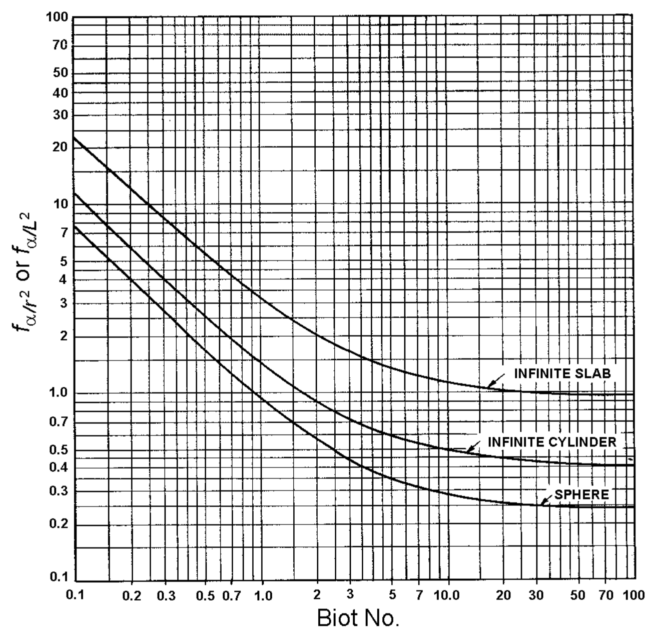

Fig. 2 Relationship Between $f\alpha/r^2$ and Biot number for Infinite Slab, Infinite Cylinder, and Sphere

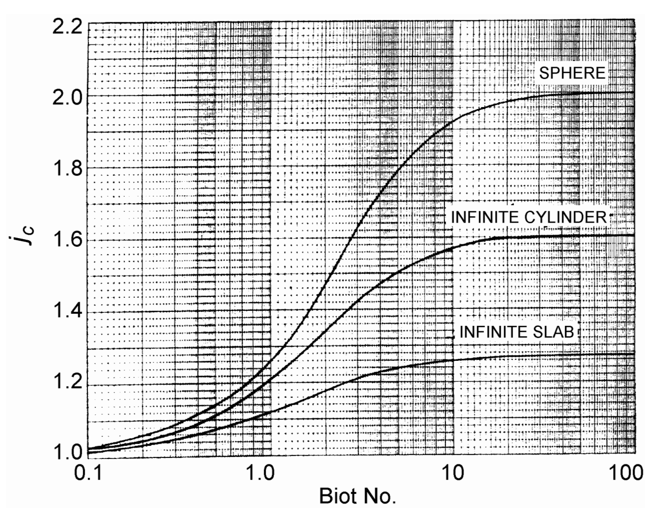

Fig. 3 Relationship Between j_c Value for Thermal Center Temperature and Biot Number for Various Shapes

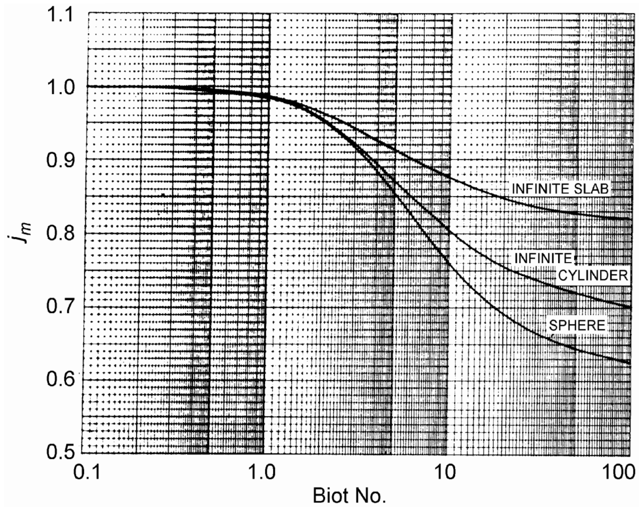

Fig. 4 Relationship Between j_m Value for Mass Average Temperature and Biot Number for Various Shapes

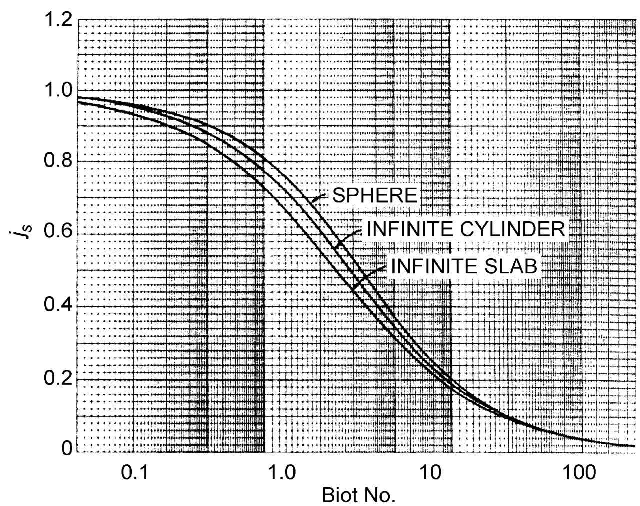

Fig. 5 Relationship Between j_s Value for Surface Temperature and Biot Number for Various Shapes

Lacroix and Castaigne (1987a) presented expressions for estimating the f and j_c factors for the thermal center temperature of infinite slabs, infinite cylinders, and spheres. These expressions, which depend upon geometry and Biot number, are summarized in Tables 1, 2 and 3. In these expressions, α is the thermal diffusivity of the food item and L is the characteristic dimension, defined to be the shortest distance from the thermal center of the food item to its surface. For an infinite slab, L is the half thickness. For an infinite cylinder or a sphere, L is the radius.

By using various combinations of infinite slabs and infinite cylinders, the f and j factors for infinite rectangular columns, finite cylinders, and rectangular bricks may be estimated. Each of these shapes can be generated by intersecting infinite slabs and infinite cylinders: two infinite slabs of proper thickness for the infinite rectangular column, one infinite slab and one infinite cylinder for the finite cylinder, or three infinite orthogonal slabs of proper thickness for the rectangular brick. The f and j factors of these composite bodies can be estimated by:

$$\frac{1}{f_{comp}} = \sum_i \left(\frac{1}{f_i}\right) \tag{6}$$

Table 1 Expressions for Estimating f and j_c Factors for the Thermal Center Temperature of Infinite Slabs

Biot Number Range	Equations for f and j factors
Bi ≤ 0.1	$\frac{f\alpha}{L^2} = \frac{\ln 10}{\text{Bi}}$ $j_c = 1.0$
0.1 < Bi ≤ 100	$\frac{f\alpha}{L^2} = \frac{\ln 10}{u^2}$ $j_c = \frac{2\sin u}{u + \sin u \cos u}$ *where* $u = 0.860972 + 0.312133\ln(\text{Bi}) + 0.007986[\ln(\text{Bi})]^2 - 0.016192[\ln(\text{Bi})]^3 - 0.001190[\ln(\text{Bi})]^4 + 0.000581[\ln(\text{Bi})]^5$
Bi > 100	$\frac{f\alpha}{L^2} = 0.9332$ $j_c = 1.273$

Source: Lacroix and Castaigne, 1987a

Table 2 Expressions for Estimating f and j_c factors for the Thermal Center Temperature of Infinite Cylinders

Biot Number Range	Equations for f and j factors
Bi ≤ 0.1	$\frac{f\alpha}{L^2} = \frac{\ln 10}{2\text{ Bi}}$ $j_c = 1.0$
0.1 < Bi ≤ 100	$\frac{f\alpha}{L^2} = \frac{\ln 10}{v^2}$ $j_c = \frac{2J_1(v)}{v[J_0^2(v) - J_1^2(v)]}$ *where* $v = 1.257493 + 0.487941\ln(\text{Bi}) + 0.025322[\ln(\text{Bi})]^2 - 0.026568[\ln(\text{Bi})]^3 - 0.002888[\ln(\text{Bi})]^4 + 0.001078[\ln(\text{Bi})]^5$ and $J_0(v)$ and $J_1(v)$ are zero and first order Bessel functions, repectively
Bi > 100	$\frac{f\alpha}{L^2} = 0.3982$ $j_c = 1.6015$

Source: Lacroix and Castaigne, 1987a

$$j_{comp} = \prod_i j_i \tag{7}$$

where the subscript i represents the appropriate infinite slab(s) or infinite cylinder. To evaluate the f_i and j_i of Equations (6) and (7), the Biot number must be defined, corresponding to the appropriate infinite slab(s) or infinite cylinder.

Determination of *f* and *j* Factors for Irregular Shapes

In an effort to determine f and j factors for irregularly shaped food items, Smith et al. (1968) developed, for the case of Biot number approaching infinity, a shape factor called the geometry index G, which is obtained as follows:

$$G = 0.25 + \frac{3}{8B_1^2} + \frac{3}{8B_2^2} \tag{8}$$

Table 3 Expressions for Estimating f and j_c factors for the Thermal Center Temperature of Spheres

Biot Number Range	Equations for f and j factors
$\text{Bi} \le 0.1$	$\frac{f\alpha}{L^2} = \frac{\ln 10}{3\ \text{Bi}}$ $j_c = 1.0$
$0.1 < \text{Bi} \le 100$	$\frac{f\alpha}{L^2} = \frac{\ln 10}{w^2}$ $j_c = \frac{2(\sin w - w\cos w)}{w - \sin w \cos w}$ *where* $w = 1.573729 + 0.642906\ln(\text{Bi}) + 0.047859[\ln(\text{Bi})]^2 - 0.03553[\ln(\text{Bi})]^3 - 0.004907[\ln(\text{Bi})]^4 + 0.001563[\ln(\text{Bi})]^5$
$\text{Bi} > 100$	$\frac{f\alpha}{L^2} = 0.2333$ $j_c = 2.0$

Source: Lacroix and Castaigne, 1987a

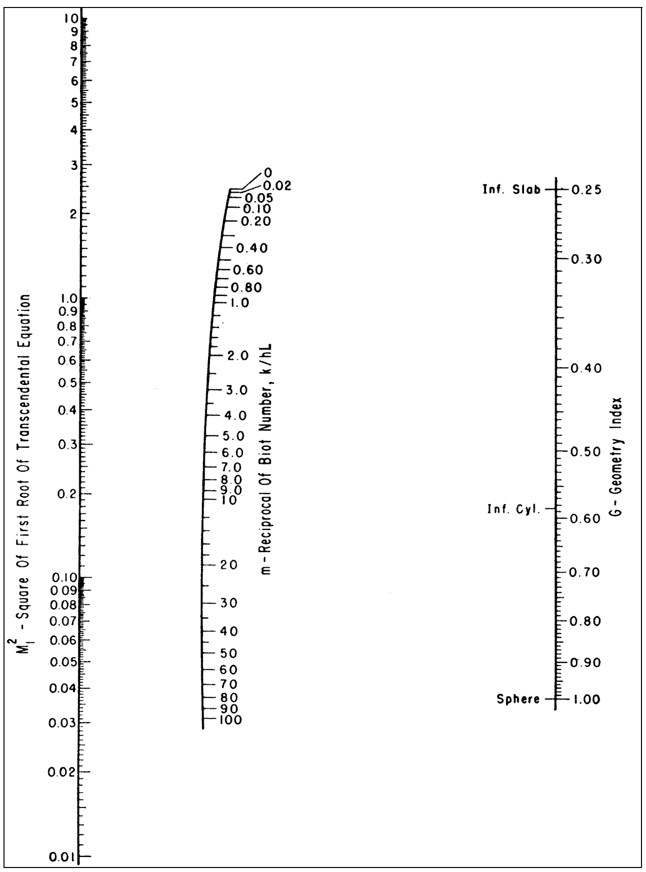

Fig. 6 Nomograph for Estimating Value of M_1^2 from Reciprocal of Biot Number and Smith's (1966) Geometry Index

where B_1 and B_2 are related to the cross sectional areas of the food:

$$B_1 = \frac{A_1}{\pi L^2} \qquad B_2 = \frac{A_2}{\pi L^2} \tag{9}$$

where L is the shortest distance between the thermal center of the food and its surface, A_1 is the minimum cross sectional area containing L and A_2 is the cross sectional area that is orthogonal to A_1.

This geometry index is used in conjunction with the inverse of the Biot number m and a nomograph (shown in Figure 6) to obtain the characteristic value $M_1{}^2$. Smith et al. showed that the characteristic value $M_1{}^2$ can be related to the f factor by

$$f = \frac{2.303L^2}{M_1^2\alpha} \tag{10}$$

where α is the thermal diffusivity of the food. In addition, an expression for estimating a j_m factor which is used to determine the mass average temperature is given as

$$j_m = 0.892e^{-0.0388M_1^2} \tag{11}$$

As an alternative to estimating the value of $M_1{}^2$ from the nomograph developed by Smith et al. (1968), Hayakawa and Villalobos (1989) obtained regression formulae for estimating $M_1{}^2$. For Biot numbers approaching infinity, their regression formula is given as follows:

$$\begin{aligned}\ln(M_1^2) = {} & 2.2893825 + 0.35330539X_g - 3.8044156X_g^2 \\ & - 9.6821811X_g^3 - 12.0321827X_g^4 \\ & - 7.1542411X_g^5 - 1.6301018X_g^6\end{aligned} \tag{12}$$

where $X_g = \ln(G)$. For finite Biot numbers, Hayakawa and Villalobos (1989) gave the following:

$$\begin{aligned}\ln(M_1^2) = {} & 0.92083090 + 0.83409615X_g - 0.78765739X_b \\ & - 0.04821784X_gX_b - 0.0408987X_g^2 \\ & - 0.10045526X_b^2 + 0.01521388X_g^3 \\ & + 0.00119941X_gX_b^3 + 0.00129982X_b^4\end{aligned} \tag{13}$$

where $X_g = \ln(G)$ and $X_b = \ln(1/\text{Bi})$.

Cooling Time Estimation Methods Based on Equivalent Heat Transfer Dimensionality

The product geometry can also be considered using a shape factor called the **equivalent heat transfer dimensionality** (Cleland and Earle (1982a), which compares the total heat transfer to the heat transfer through the shortest dimension. Cleland and Earle developed an expression for estimating the equivalent heat transfer dimensionality of irregularly shaped food items as a function of Biot number. This overcomes the limitation of the geometry index developed by Smith et al. (1968), which was derived for the case of Biot number approaching infinity. However, the cooling time estimation method developed by Cleland and Earle, requires the use of a nomograph. Lin et al. (1993, 1996a, 1996b) expanded on the work of Cleland and Earle to eliminate the need for a nomograph.

In the method of Lin et al., the cooling time of a food or beverage is estimated by a first term approximation to the analytical solution for convective cooling of a sphere:

$$\theta = \frac{3\rho cL^2}{\omega^2 kE}\ln\left(\frac{j}{Y}\right) \tag{14}$$

where θ is the cooling time, ρ is the density of the food, c is the specific heat of the food, L is the radius or half-thickness of the food item, k is the thermal conductivity of the food, j is the previously discussed lag factor, E is the equivalent heat transfer dimensionality and ω is the first root (in radians) of the following transcendental function:

$$\omega \cot \omega + \mathrm{Bi} - 1 = 0 \qquad (15)$$

In Equation (14), the equivalent heat transfer dimensionality E is given as a function of Biot number:

$$E = \frac{\mathrm{Bi}^{4/3} + 1.85}{\dfrac{\mathrm{Bi}^{4/3}}{E_\infty} + \dfrac{1.85}{E_o}} \qquad (16)$$

E_o and E_∞ are the equivalent heat transfer dimensionalities for the limiting cases of Bi = 0 and $\mathrm{Bi} \to \infty$, respectively. The definitions of E_o and E_∞ make use of the dimensional ratios β_1 and β_2:

$$\beta_1 = \frac{\text{second shortest dimension of food item}}{\text{shortest dimension of food item}} \qquad (17)$$

$$\beta_2 = \frac{\text{longest dimension of food item}}{\text{shortest dimension of food item}} \qquad (18)$$

For two-dimensional, irregularly shaped food items, E_o, which is the equivalent heat transfer dimensionality for Bi = 0, is given by:

$$E_o = \left(1 + \frac{1}{\beta_1}\right)\left[1 + \left(\frac{\beta_1 - 1}{2\beta_1 + 2}\right)^2\right] \qquad (19)$$

For three-dimensional, irregularly shaped food items, E_o is:

$$E_o = 1.5\frac{\beta_1 + \beta_2 + \beta_1^2(1+\beta_2) + \beta_2^2(1+\beta_1)}{\beta_1\beta_2(1+\beta_1+\beta_2)} - \frac{[(\beta_1 - \beta_2)^2]^{0.4}}{15} \qquad (20)$$

For finite cylinders, bricks and infinite rectangular rods, E_o may be determined as follows:

$$E_o = 1 + \frac{1}{\beta_1} + \frac{1}{\beta_2} \qquad (21)$$

For spheres, infinite cylinders, and infinite slabs, $E_o = 3$, $E_o = 2$, and $E_o = 1$, respectively.

For both two-dimensional and three-dimensional food items, the general form for E_o at $\mathrm{Bi} \to \infty$ is given as:

$$E_\infty = 0.75 + p_1 f(\beta_1) + p_2 f(\beta_2) \qquad (22)$$

where

$$f(\beta) = \frac{1}{\beta^2} + 0.01 p_3 \exp\left[\beta - \frac{\beta^2}{6}\right] \qquad (23)$$

with β_1 and β_2 as previously defined. The geometric parameters, p_1, p_2, and p_3, are given in Table 4 for various geometries.

Lin et al. (1993, 1996a, 1996b) also developed an expression for the lag factor applicable to the thermal center of a food item j_c as:

$$j_c = \frac{\mathrm{Bi}^{1.35} + \dfrac{1}{\lambda}}{\dfrac{\mathrm{Bi}^{1.35}}{L_\infty} + \dfrac{1}{\lambda}} \qquad (24)$$

Table 4 Geometric Parameters

Shape	N	p_1	p_2	p_3	γ_1	γ_2	λ
Infinite slab ($\beta_1 = \beta_2 = \infty$)	1	0	0	0	∞	∞	1
Infinite rectangular rod ($\beta_1 \ge 1$, $\beta_2 = \infty$)	2	0.75	0	−1	$4\beta_1/\pi$	∞	γ_1
Brick ($\beta_1 \ge 1$, $\beta_2 \ge \beta_1$)	3	0.75	0.75	−1	$4\beta_1/\pi$	$1.5\beta_2$	γ_1
Infinite Cylinder ($\beta_1 = 1$, $\beta_2 = \infty$)	2	1.01	0	0	1	∞	1
Infinite ellipse ($\beta_1 > 1$, $\beta_2 = \infty$)	2	1.01	0	1	β_1	∞	γ_1
Squat cylinder ($\beta_1 = \beta_2$, $\beta_1 \ge 1$)	3	1.01	0.75	−1	$1.225\beta_1$	$1.225\beta_2$	γ_1
Short cylinder ($\beta_1 = 1$, $\beta_2 \ge 1$)	3	1.01	0.75	−1	β_1	$1.5\beta_2$	γ_1
Sphere ($\beta_1 = \beta_2 = 1$)	3	1.01	1.24	0	1	1	1
Ellipsoid ($\beta_1 \ge 1$, $\beta_2 \ge \beta_1$)	3	1.01	1.24	1	β_1	β_2	γ_1

Source: Lin et al. (1996b)

where L_∞ is as follows:

$$L_\infty = 1.271 + 0.305\exp(0.172\gamma_1 - 0.115\gamma_1^2) + 0.425\exp(0.09\gamma_2 - 0.128\gamma_2^2) \qquad (25)$$

and the geometric parameters λ, γ_1 and γ_2 are given in Table 4.

For the mass average temperature, Lin et al. gave the lag factor j_m as follows:

$$j_m = \mu j_c \qquad (26)$$

where

$$\mu = \left[\frac{1.5 + 0.69\mathrm{Bi}}{1.5 + \mathrm{Bi}}\right]^N \qquad (27)$$

and N is the number of dimensions of a food item in which heat transfer is significant. The value of N for various geometries is also given in Table 4.

Algorithms for Estimating Cooling Time

The following suggested algorithm for estimating cooling time of foods and beverages is based on the equivalent heat transfer dimensionality method presented by Lin et al. (1993, 1996a, 1996b).

1. Determine thermal properties of the food item (see Chapter 8).
2. Determine surface heat transfer coefficient for the cooling process (see Chapter 8).
3. Determine characteristic dimension L and dimensional ratios β_1 and β_2 using Equations (17) and (18).
4. Calculate Biot number using Equation (2).
5. Calculate equivalent heat transfer dimensionality E for food geometry using Equation (16). This calculation requires evaluation of E_o and E_∞ using Equations (19) through (23).
6. Calculate lag factor corresponding to thermal center and/or mass average of food item using Equations (24) through (27).
7. Calculate root of transcendental equation given in Equation (15).
8. Calculate cooling time using Equation (14).

The following alternative algorithm for estimating the cooling time of foods and beverages is based on the use of f and j factors.

1. Determine thermal properties of food item (see Chapter 8).
2. Determine surface heat transfer coefficient for cooling process (see Chapter 8).
3. Determine characteristic dimension L of food item.
4. Calculate Biot number using Equation (2).
5. Calculate f and j factors by one of the following methods:
   (a) Method of Pflug et al. (1965): Figures 2 through 5.
   (b) Method of Lacroix and Castaigne (1987a): Tables 1, 2, and 3.
   (c) Method of Smith et al. (1968): Equations (8) through (11) and Figure 6.
   (d) Method of Hayakawa and Villalobos (1989): Equations (12) and (13) in conjunction with Equations (8) through (11).
6. Calculate cooling time using Equation (4).

SAMPLE PROBLEMS FOR ESTIMATING COOLING TIME

Example 1. A piece of ham, initially at 70°C, is to be cooled in an airblast freezer. The air temperature within the freezer is −1°C and the surface heat transfer coefficient is estimated to be 48.0 W/(m^2·K). The overall dimension of the ham is 0.102 m by 0.165 m by 0.279 m. Estimate the time required for the mass average temperature of the ham to reach 10°C. The thermophysical properties for ham are given as follows:

c = 3740 J/(kg·K)
k = 0.379 W/(m·K)
ρ = 1080 kg/m^3

Solution: Use the algorithm based on the method of Lin et al. (1993, 1996a, 1996b).

Step 1: Determine the thermal properties (c, k, ρ) of the ham.
In this example the thermal properties of the ham are given above.

Step 2: Determine the heat transfer coefficient h.
The heat transfer coefficient is given as h = 48.0 W/(m^2·K)

Step 3: Determine the characteristic dimension L and the dimensional ratios β_1 and β_2.
For cooling time problems, the characteristic dimension is the shortest distance from the thermal center of a food item to its surface. Assuming that the thermal center of the piece of ham coincides with its geometric center, the characteristic dimension becomes:

$$L = (0.102\text{ m})/2 = 0.051\text{ m}$$

The dimensional ratios then become [Equations (17) and (18)]:

$$\beta_1 = \frac{0.165\text{ m}}{0.102\text{ m}} = 1.62$$

$$\beta_2 = \frac{0.279\text{ m}}{0.102\text{ m}} = 2.74$$

Step 4: Calculate the Biot number.

$$\text{Bi} = hL/k = (48.0)(0.051)/0.379 = 6.46$$

Step 5: Calculate the heat transfer dimensionality.
Using Equation (20), E_o becomes:

$$E_o = \frac{1.5[1.62 + 2.74 + 1.62^2(1 + 2.74) + 2.74^2(1 + 1.62)]}{(1.62)(2.74)(1 + 1.62 + 2.74)} - \frac{[(1.62 - 2.74)^2]^{0.4}}{15} = 2.06$$

Assuming the ham to be ellipsoidal, the geometric factors can be obtained from Table 4:

$p_1 = 1.01$ $p_2 = 1.24$ $p_3 = 1$

From Equation (23)

$$f(\beta_1) = \frac{1}{1.62^2} + (0.01)(1)\exp\left[1.62 - \frac{1.62^2}{6}\right] = 0.414$$

$$f(\beta_2) = \frac{1}{2.74^2} + (0.01)(1)\exp\left[2.74 - \frac{2.74^2}{6}\right] = 0.178$$

From Equation (22):

$$E_\infty = 0.75 + (1.01)(0.414) + (1.24)(0.178) = 1.39$$

Thus, using Equation (16), the equivalent heat transfer dimensionality becomes:

$$E = \frac{6.46^{4/3} + 1.85}{\frac{6.46^{4/3}}{1.39} + \frac{1.85}{2.06}} = 1.45$$

Step 6: Calculate the lag factor applicable to the mass average temperature.
From Table 4, $\lambda = \beta_1$, $\gamma_1 = \beta_1$ and $\gamma_2 = \beta_2$. Using Equation (25), L_∞ becomes:

$$L_\infty = 1.271 + 0.305\exp[(0.172)(1.62) - (0.115)(1.62)^2] + 0.425\exp[(0.09)2.74 - (0.128)(2.74)^2] = 1.78$$

Using Equation (24), the lag factor applicable to the center temperature becomes:

$$j_c = \frac{6.46^{1.35} + \frac{1}{1.62}}{\frac{6.46^{1.35}}{1.78} + \frac{1}{1.62}} = 1.72$$

Using Equations (26) and (27), the lag factor for the mass average temperature becomes:

$$j_m = \left[\frac{1.5 + (0.69)(6.46)}{1.5 + 6.46}\right]^3 (1.72) = 0.721$$

Step 7: Find the root of transcendental Equation (15):

$$\omega\cot\omega + \text{Bi} - 1 = 0$$

$$\omega\cot\omega + 6.46 - 1 = 0$$

$$\omega = 2.68$$

Step 8: Calculate the cooling time.
The unaccomplished temperature difference is:

$$Y = \frac{T_m - T}{T_m - T_i} = \frac{-1 - 10}{-1 - 70} = 0.1549$$

Using Equation (14), the cooling time becomes

$$\theta = \frac{(3)(1080)(3740)(0.051)^2}{(2.68)^2(0.379)(1.45)}\ln\left[\frac{0.721}{0.1549}\right]$$

$$\theta = 12\,280\text{ s} = 3.41\text{ h}$$

Example 2. Repeat the cooling time calculation of Example 1, but use the cooling time estimation algorithm based on the use of f and j factors.

Solution: Use the algorithm based on the method of Hayakawa and Villalobos (1989) for f and j factors.

Step 1: Determine the thermal properties of the ham.
The thermal properties of the ham are given in Example 1.

Step 2: Determine the heat transfer coefficient.
The heat transfer coefficient is given from Example 1 as

$$h = 48.0\text{ W/(m}^2\cdot\text{K)}$$

Step 3: Determine the characteristic dimension L and the dimensional ratios β_1 and β_2.
From Example 1, L = 0.051 m, β_1 = 1.62, β_2 = 2.74

Step 4: Calculate the Biot number.

From Example 1, Bi = 6.46.

Step 5: Calculate the *f* and *j* factors using the method of Hayakawa and Villalobos (1989).

For simplicity, assume the cross-sections of the ham to be ellipsoidal in shape. The area of an ellipse is the product of π times the minor axis times the major axis, or:

$$A_1 = \pi L^2 \beta_1 \qquad A_2 = \pi(\beta_1 L)^2 \frac{\beta_2}{\beta_1}$$

Using Equations (8) and (9), calculate the geometry index G:

$$B_1 = \frac{A_1}{\pi L^2} = \frac{\pi L^2 \beta_1}{\pi L^2} = \beta_1 = 1.62$$

$$B_2 = \frac{A_2}{\pi L^2} = \frac{\pi(\beta_1 L)^2 \beta_2}{\pi L^2 \beta_1} = \beta_1 \beta_2 = (1.62)(2.74) = 4.44$$

$$G = 0.25 + \frac{3}{(8)(1.62)^2} + \frac{3}{(8)(4.44)^2} = 0.412$$

Using Equation (13), determine the characteristic value M_1^2:

$$X_g = \ln(G) = \ln(0.412) = -0.887$$

$$X_b = \ln(1/\text{Bi}) = \ln(1/6.46) = -1.87$$

$$\begin{aligned}\ln(M_1^2) = {} & 0.92083090 + 0.83409615(-0.887) - 0.78765739(-1.87) \\ & - 0.04821784(-0.887)(-1.87) - 0.0408987(-0.887)^2 \\ & - 0.10045526(-1.87)^2 + 0.01521388(-0.887)^3 \\ & + 0.00119941(-0.887)(-1.87)^3 \\ & + 0.00129982(-1.87)^4 = 1.20\end{aligned}$$

$$M_1^2 = 3.32$$

From Equation (10), the *f* factor becomes:

$$f = \frac{2.303 L^2}{M_1^2 \alpha} = \frac{2.303 L^2 \rho c}{M_1^2 k}$$

$$f = \frac{(2.303)(0.051)^2(1.08\times10^6)(3.74)}{(3.32)(0.379)} = 19230 \text{ s} = 5.24 \text{ h}$$

From Equation (11), the *j* factor becomes:

$$j_m = 0.892 e^{(-0.0388)(3.32)} = 0.784$$

Step 6: Calculate the cooling time.

From Example 1, the unaccomplished temperature difference was found to be $Y = 0.1549$. Using Equation (5), the cooling time becomes:

$$\theta = -\frac{19230}{2.303} \ln\left(\frac{0.1549}{0.784}\right) = 13500 \text{ s} = 3.75 \text{ h}$$

FREEZING TIMES OF FOODS AND BEVERAGES

As discussed at the beginning of this chapter, the freezing of foods and beverages is not an isothermal process but rather occurs over a range of temperatures. In the following section, the basic freezing time estimation method developed by Plank is discussed first, followed by a discussion of those methods that modify Plank's equation. The discussion then focuses on those freezing time estimation methods in which the freezing time is calculated as the sum of the precooling, phase change, and subcooling times. The last section deals with freezing time estimation methods for irregularly shaped food items. These methods are divided into three subgroups: (1) equivalent heat transfer dimensionality, (2) mean conducting path, and (3) equivalent sphere diameter.

Plank's Equation

One of the most widely known simple methods for estimating freezing times of foods and beverages is that developed by Plank (1913, 1941). In this method, convective heat transfer is assumed to occur between the food item and the surrounding cooling medium. In addition, the temperature of the food item is assumed to be at its initial freezing temperature and that this temperature is constant throughout the freezing process. Furthermore, a constant thermal conductivity for the frozen region is assumed. Plank's freezing time estimation is as follows:

$$\theta = \frac{L_f}{T_f - T_m}\left[\frac{PD}{h} + \frac{RD^2}{k_s}\right] \tag{28}$$

where L_f is the volumetric latent heat of fusion, T_f is the initial freezing temperature of the food, T_m is the freezing medium temperature, D is the thickness of the slab or the diameter of the sphere or infinite cylinder, h is the convective heat transfer coefficient, k_s is the thermal conductivity of the fully frozen food, and P and R are geometric factors. For the infinite slab, $P = 1/2$ and $R = 1/8$. For a sphere, $P = 1/6$ and $R = 1/24$; and for an infinite cylinder, $P = 1/4$ and $R = 1/16$.

Plank's geometric factors indicate that an infinite slab of thickness D, an infinite cylinder of diameter D and a sphere of diameter D, if exposed to the same conditions, would have freezing times in the ratio of 6:3:2. Hence, a cylinder freezes in half the time of a slab and a sphere freezes in one-third the time of a slab.

Modifications to Plank's Equation

Various researchers have noted that Plank's method does not accurately predict freezing times of foods and beverages. This is due, in part, to the fact that Plank's method assumes that freezing of foods takes place at a constant temperature, and not over a range of temperatures as is the case in actual food freezing processes. In addition, the thermal conductivity of the frozen food is assumed to be constant; but in reality, the thermal conductivity varies greatly during freezing. Another limitation of Plank's equation is that it neglects the removal of sensible heat above the freezing point, and thus, precooling times must be calculated by using one of the methods discussed in the Cooling Times of Foods and Beverages section of this chapter. Furthermore, Plank's method only applies to infinite slabs, infinite cylinders, and spheres. Subsequently, researchers have developed improved semi-analytical/empirical cooling and freezing time estimation methods that account for precooling and subcooling times, non-constant thermal properties, irregular geometries, and phase change over a range of temperatures.

Cleland and Earle (1977, 1979a,b) incorporated corrections to account for the removal of sensible heat both above and below the initial freezing point of the food as well as temperature variation during freezing. Regression equations were developed to estimate the geometric parameters P and R for infinite slabs, infinite cylinders, spheres, and rectangular bricks. In these regression equations, the effects of surface heat transfer, precooling, and final subcooling are accounted for by means of the Biot number, the Plank number, and the Stefan number, respectively.

In this section the Biot number is defined as

$$\text{Bi} = \frac{hD}{k} \tag{29}$$

where h is the convective heat transfer coefficient, D is the characteristic dimension, and k is the thermal conductivity. The characteristic dimension D is defined to be twice the shortest distance from the thermal center of a food item to its surface. For an infinite slab, D is the thickness. For an infinite cylinder or a sphere, D is the diameter.

In general, the Plank number is defined as follows:

$$\text{Pk} = \frac{C_l(T_i - T_f)}{\Delta H} \quad (30)$$

where C_l is the volumetric specific heat of the unfrozen phase and ΔH is the volumetric enthalpy change of the food between T_f and the final food temperature. The Stefan number is similarly defined as

$$\text{Ste} = \frac{C_s(T_f - T_m)}{\Delta H} \quad (31)$$

where C_s is the volumetric specific heat of the frozen phase.

In Cleland and Earle's method, Plank's original geometric factors P and R are replaced with the modified values given in Table 5, and the latent heat L_f is replaced with the volumetric enthalpy change of the food ΔH_{10} between the freezing temperature T_f and the final center temperature, assumed to be −10°C. As shown in Table 5, the geometric factors P and R are functions of the Plank number and the Stefan number. Both of these parameters should be evaluated using the enthalpy change ΔH_{10}. Thus, the modified Plank equation takes the form:

$$\theta = \frac{\Delta H_{10}}{T_f - T_m}\left[\frac{PD}{h} + \frac{RD^2}{k_s}\right] \quad (32)$$

where k_s is the thermal conductivity of the fully frozen food.

Equation (32) is based on curve-fitting of experimental data in which the product final center temperature was −10°C. Cleland and Earle (1984) noted that this prediction formula does not perform as well when applied to situations with final center temperatures other than −10°C. Cleland and Earle proposed the following modified form of Equation (32) to account for different final center temperatures:

$$\theta = \frac{\Delta H_{10}}{T_f - T_m}\left(\frac{PD}{h} + \frac{RD^2}{k_s}\right)\left[1 - \frac{1.65\ \text{Ste}}{k_s}\ln\left(\frac{T_c - T_m}{T_{ref} - T_m}\right)\right] \quad (33)$$

where T_{ref} is −10°C, T_c is the final product center temperature, and ΔH_{10} is the volumetric enthalpy difference between the initial freezing temperature T_f and −10°C. The values of P, R, Pk, and Ste should be evaluated using ΔH_{10}, as previously discussed.

Hung and Thompson (1983) also improved upon Plank's equation to develop an alternative freezing time estimation method for infinite slabs. Their equation incorporates the volumetric change in enthalpy ΔH_{18} for the freezing process as well as a weighted average temperature difference between the initial temperature of the food and the freezing medium temperature. This weighted average temperature difference ΔT is given as follows:

$$\Delta T = (T_f - T_m) + \frac{(T_i - T_f)^2\frac{C_l}{2} - (T_f - T_c)^2\frac{C_s}{2}}{\Delta H_{18}} \quad (34)$$

where T_c is the final center temperature of the food and ΔH_{18} is the enthalpy change of the food between the initial temperature and the final center temperature, assumed to be −18°C. Empirical equations were developed to estimate P and R for infinite slabs as follows:

$$P = 0.7306 - 1.083\ \text{Pk} + \text{Ste}\left(15.40U - 15.43 + 0.01329\frac{\text{Ste}}{\text{Bi}}\right) \quad (35)$$

$$R = 0.2079 - 0.2656U(\text{Ste}) \quad (36)$$

where $U = \Delta T/(T_f - T_m)$. In these expressions Pk and Ste should be evaluated using the enthalpy change ΔH_{18}. The freezing time prediction model is:

$$\theta = \frac{\Delta H_{18}}{\Delta T}\left[\frac{PD}{h} + \frac{RD^2}{k_s}\right] \quad (37)$$

Cleland and Earle (1984) applied a correction factor to the Hung and Thompson model [Equation (37)] and improved the prediction accuracy of the model for final temperatures other than −18°C. The correction to Equation (37) is as follows:

$$\theta = \frac{\Delta H_{18}}{\Delta T}\left(\frac{PD}{h} + \frac{RD^2}{k_s}\right)\left[1 - \frac{1.65\ \text{Ste}}{k_s}\ln\left(\frac{T_c - T_m}{T_{ref} - T_m}\right)\right] \quad (38)$$

where T_{ref} is −18°C, T_c is the product final center temperature and ΔH_{18} is the volumetric enthalpy change between the initial temperature T_i and −18°C. The weighted average temperature difference ΔT, Pk, and Ste should be evaluated using ΔH_{18}.

Precooling, Phase Change, and Subcooling Time Calculations

Total freezing time θ is as follows:

$$\theta = \theta_1 + \theta_2 + \theta_3 \quad (39)$$

where θ_1, θ_2, and θ_3 are the precooling, phase change, and subcooling times, respectively.

DeMichelis and Calvelo (1983) suggested that the equivalent heat transfer dimensionality method of Cleland and Earle (1982), discussed in the Cooling Times of Foods and Beverages section of this chapter, be used to estimate the precooling and subcooling times of foods and beverages. They also suggested that the phase change time calculation be made with Plank's equation. However, they suggested that in Plank's equation, the thermal conductivity of the frozen food be evaluated at the temperature, $(T_f + T_m)/2$, where T_f is the initial freezing temperature of the food and T_m is the temperature of the cooling medium. The use of Plank's equation limits the applicability of this method to infinite slabs, infinite cylinders and spheres.

Lacroix and Castaigne (1987a, 1987b, 1988) suggested the use of f and j factors to determine precooling and subcooling times of foods and beverages. They presented equations, given in Tables 1, 2, and 3, for estimating the values of f and j for infinite slabs, infinite cylinders, and spheres. Note that Lacroix and Castaigne based the Biot number on the shortest distance between the thermal center of the food item and its surface—not twice that distance.

Lacroix and Castaigne (1987a, 1987b, 1988) gave the following expression for estimating the precooling time θ_1:

$$\theta_1 = f_1 \log\left[j_1\frac{T_m - T_i}{T_m - T_f}\right] \quad (40)$$

where T_m is the coolant temperature, T_i is the initial temperature of the food, and T_f is the initial freezing point of the food. The f_1 and j_1 factors are determined from a Biot number that is calculated using an average thermal conductivity. This average thermal conductivity is based on the thermal conductivity of the unfrozen food, and the thermal conductivity of the frozen food evaluated at $(T_f + T_m)/2$.

The expression for estimating the subcooling time θ_3 is:

$$\theta_3 = f_3 \log\left[j_3 \frac{T_m - T_f}{T_m - T_c}\right] \qquad (41)$$

where T_c is the final temperature at the center of the food item. The f_3 and j_3 factors are determined from a Biot number that is calculated using the thermal conductivity of the frozen food evaluated at the temperature $(T_f + T_m)/2$.

Lacroix and Castaigne model the phase change time θ_2 with Plank's equation:

$$\theta_2 = \frac{L_f D^2}{(T_f - T_m)k_c}\left[\frac{P}{2\text{Bi}_c} + R\right] \qquad (42)$$

where L_f is the volumetric latent heat of fusion of the food, P and R are the original Plank geometric shape factors, k_c is the thermal conductivity of the frozen food at $(T_f + T_m)/2$, and Bi_c is the Biot number for the subcooling period ($\text{Bi}_c = hL/k_c$).

Lacroix and Castaigne (1987a, 1987b) adjusted P and R to obtain better agreement between predicted freezing times and experimental data. Using regression analysis, Lacroix and Castaigne suggested the following geometric factors:

For **infinite slabs**:

$$P = 0.51233 \qquad (43)$$

$$R = 0.15396 \qquad (44)$$

For **infinite cylinders**:

$$P = 0.27553 \qquad (45)$$

$$R = 0.07212 \qquad (46)$$

For **spheres**:

$$P = 0.19665 \qquad (47)$$

$$R = 0.03939 \qquad (48)$$

For **rectangular bricks**:

$$P = P'\left[-0.02175\frac{1}{\text{Bi}_c} - 0.01956\frac{1}{\text{Ste}} - 1.69657\right] \qquad (49)$$

$$R = R'\left[5.57519\frac{1}{\text{Bi}_c} + 0.02932\frac{1}{\text{Ste}} + 1.58247\right] \qquad (50)$$

For rectangular bricks, the values for P' and R' are calculated using the expressions given in Table 5 for the P and R of bricks.

Pham (1984) also devised a freezing time estimation method, similar to Plank's equation, in which sensible heat effects were considered by calculating precooling, phase change, and subcooling times separately. In addition, Pham suggested the use of a mean freezing point, which is assumed to be 1.5 K below the initial freezing point of the food, to account for freezing that takes place over a range of temperatures. Pham's freezing time estimation method is stated in terms of the volume and surface area of the food item and is, therefore, applicable to food items of any shape. This method is given as:

$$\theta_i = \frac{Q_i}{hA_s \Delta T_{mi}}\left(1 + \frac{\text{Bi}_i}{k_i}\right) \qquad i = 1, 2, 3 \qquad (51)$$

where θ_1 is the precooling time, θ_2 is the phase change time, θ_3 is the subcooling time, and the remaining variables are defined as shown in Table 6.

Pham (1986a) significantly simplified the previous freezing time estimation method to yield:

$$\theta = \frac{V}{hA_s}\left(\frac{\Delta H_1}{\Delta T_1} + \frac{\Delta H_2}{\Delta T_2}\right)\left(1 + \frac{\text{Bi}_s}{4}\right) \qquad (52)$$

in which

$$\begin{aligned} \Delta H_1 &= C_l(T_i - T_{fm}) \\ \Delta H_2 &= L_f + C_s(T_{fm} - T_c) \end{aligned} \qquad (53)$$

$$\begin{aligned} \Delta T_1 &= \frac{T_i + T_{fm}}{2} - T_m \\ \Delta T_2 &= T_{fm} - T_m \end{aligned} \qquad (54)$$

where C_l and C_s are volumetric specific heats above and below freezing, respectively, T_i is the initial food temperature, L_f is the volumetric latent heat of freezing, and V is the volume of the food item.

Pham suggested that the mean freezing temperature T_{fm} used in Equations (53) and (54) mainly depended on the cooling medium temperature T_m and the product center temperature T_c. By curve fitting to existing experimental data, Pham (1986a) proposed the following equation to determine the mean freezing temperature for use in Equations (53) and (54):

$$T_{fm} = 1.8 + 0.26T_c + 0.105T_m \qquad (55)$$

where all temperatures are in °C.

Pham (1986b) subsequently extended the applicability of the simplified freezing time estimation model by providing provisions for variations in environmental conditions. Equations were presented to calculate the location of the freezing front for cases when step changes in boundary conditions exist. In addition, Pham (1987) developed equations to predict the location of the freezing front for the case of asymmetric heat transfer coefficients and asymmetric cooling medium temperatures.

Geometric Considerations

Equivalent Heat Transfer Dimensionality. Similar to their work involving cooling times of foods, Cleland and Earle (1982b) also introduced a geometric correction factor, called the **equivalent heat transfer dimensionality** E to calculate the freezing times of irregularly shaped food items. The freezing time of an irregularly shaped object θ_{shape}, was related to the freezing time of an infinite slab θ_{slab} via the equivalent heat transfer dimensionality as:

$$\theta_{shape} = \theta_{slab}/E \qquad (56)$$

The freezing time of the infinite slab is then calculated from one of the many suitable freezing time estimation methods available for infinite slabs.

Using data collected from a large number of freezing experiments, Cleland and Earle (1982) developed empirical correlations for the equivalent heat transfer dimensionality applicable to rectangular bricks and finite cylinders. For rectangular brick shapes with dimensions D by $\beta_1 D$ by $\beta_2 D$, the equivalent heat transfer dimensionality was given as follows:

Table 5 Expressions for *P* and *R*

Shape	*P* and *R* Expressions
Infinite slab	$P = 0.5072 + 0.2018\ \mathrm{Pk} + \mathrm{Ste}\left[0.3224\ \mathrm{Pk} + \frac{0.0105}{\mathrm{Bi}} + 0.0681\right]$
	$R = 0.1684 + \mathrm{Ste}(0.2740\ \mathrm{Pk} - 0.0135)$
Infinite cylinder	$P = 0.3751 + 0.0999\ \mathrm{Pk} + \mathrm{Ste}\left[0.4008\ \mathrm{Pk} + \frac{0.0710}{\mathrm{Bi}} - 0.5865\right]$
	$R = 0.0133 + \mathrm{Ste}(0.0415\ \mathrm{Pk} + 0.3957)$
Sphere	$P = 0.1084 + 0.0924\ \mathrm{Pk} + \mathrm{Ste}\left[0.231\ \mathrm{Pk} - \frac{0.3114}{\mathrm{Bi}} + 0.6739\right]$
	$R = 0.0784 + \mathrm{Ste}(0.0386\ \mathrm{Pk} - 0.1694)$
Brick	$P = P_2 + P_1[0.1136 + \mathrm{Ste}(5.766P_1 - 1.242)]$
	$R = R_2 + R_1[0.7344 + \mathrm{Ste}(49.89R_1 - 2.900)]$
	where
	$P_2 = P_1\left[1.026 + 0.5808\mathrm{Pk} + \mathrm{Ste}\left(0.2296\mathrm{Pk} + \frac{0.0182}{\mathrm{Bi}} + 0.1050\right)\right]$
	$R_2 = R_1[1.202 + \mathrm{Ste}(3.410\mathrm{Pk} + 0.7336)]$
	and
	$P_1 = \frac{\beta_1\beta_2}{2(\beta_1\beta_2 + \beta_1 + \beta_2)}$
	$R_1 = \frac{Q}{2}\left[(r-1)(\beta_1 - r)(\beta_2 - r)\ln\left(\frac{r}{r-1}\right) - (s-1)(\beta_1 - s)(\beta_2 - s)\ln\left(\frac{s}{s-1}\right)\right] + \frac{1}{72}(2\beta_1 + 2\beta_2 - 1)$
	in which
	$\frac{1}{Q} = 4[(\beta_1 - \beta_2)(\beta_1 - 1) + (\beta_2 - 1)^2]^{1/2}$
	$r = \frac{1}{3}\{\beta_1 + \beta_2 + 1 + [(\beta_1 - \beta_2)(\beta_1 - 1) + (\beta_2 - 1)^2]^{1/2}\}$
	$s = \frac{1}{3}\{\beta_1 + \beta_2 + 1 - [(\beta_1 - \beta_2)(\beta_1 - 1) + (\beta_2 - 1)^2]^{1/2}\}$

Cleland and Earle (1977, 1979a, 1979b)

$$E = 1 + W_1 + W_2 \tag{57}$$

where

$$W_1 = \left(\frac{\mathrm{Bi}}{\mathrm{Bi} + 2}\right)\frac{5}{8\beta_1^3} + \left(\frac{2}{\mathrm{Bi} + 2}\right)\frac{2}{\beta_1(\beta_1 + 1)} \tag{58}$$

and

$$W_2 = \left(\frac{\mathrm{Bi}}{\mathrm{Bi} + 2}\right)\frac{5}{8\beta_2^3} + \left(\frac{2}{\mathrm{Bi} + 2}\right)\frac{2}{\beta_2(\beta_2 + 1)} \tag{59}$$

For the case of finite cylinders where the diameter is smaller than the height, the equivalent heat transfer dimensionality was given as:

$$E = 2.0 + W_2 \tag{60}$$

In addition, Cleland et al. (1987a, 1987b) developed expressions for determining the equivalent heat transfer dimensionality of infinite slabs, infinite and finite cylinders, rectangular bricks, spheres, and two- and three-dimensional irregular shapes. Numerical methods were used to calculate the freezing or thawing times for these various shapes. A non-linear regression analysis of the resulting these series expressions is significant. The resulting expressions for numerical data yielded the following form for the equivalent heat transfer dimensionality:

$$E = G_1 + G_2E_1 + G_3E_2 \tag{61}$$

where

$$E_1 = X(2.32/\beta_1^{1.77})\frac{1}{\beta_1} + [1 - X(2.32/\beta_1^{1.77})]\frac{0.73}{\beta_1^{2.50}} \tag{62}$$

$$E_2 = X(2.32/\beta_2^{1.77})\frac{1}{\beta_2} + [1 - X(2.32/\beta_2^{1.77})]\frac{0.73}{\beta_2^{2.50}} \tag{63}$$

$$X(x) = x/(\mathrm{Bi}^{1.34} + x) \tag{64}$$

where the geometric constants G_1, G_2, and G_3 are given in Table 7.

Using the freezing time prediction methods for infinite slabs and various multi-dimensional shapes developed by McNabb et al. (1990), Hossain et al. (1992a) derived infinite series expressions for E of infinite rectangular rods, finite cylinders, and rectangular bricks. For most practical freezing situations, only the first term of E are given in Table 8.

Table 6 Definition of Variables for the Freezing Time Estimation Method

Process	Variables
Precooling	$i = 1$ $k_1 = 6$ $Q_1 = C_l(T_i - T_{fm})V$ $\mathrm{Bi}_1 = (\mathrm{Bi}_l + \mathrm{Bi}_s)/2$ $\Delta T_{m1} = \dfrac{(T_i - T_m) - (T_{fm} - T_m)}{\ln\left[\dfrac{T_i - T_m}{T_{fm} - T_m}\right]}$
Phase change	$i = 2$ $k_2 = 4$ $Q_3 = C_s(T_{fm} - T_c)V$ $Q_2 = L_f V$ $\mathrm{Bi}_2 = \mathrm{Bi}_s$ $\Delta T_{m2} = T_{fm} - T_m$
Subcooling	$i = 3$ $k_3 = 6$ $\mathrm{Bi}_3 = \mathrm{Bi}_s$ $\Delta T_{m3} = \dfrac{(T_{fm} - T_m) - (T_o - T_m)}{\ln\left[\dfrac{T_{fm} - T_m}{T_o - T_m}\right]}$

Source: Pham (1984)
Notes:
A_s = area through which heat is transferred
Bi_l = Biot number for unfrozen phase
Bi_s = Biot number for frozen phase
Q_1, Q_2, Q_3 = heats of precooling, phase change, and subcooling, respectively
$\Delta T_{m1}, \Delta T_{m2}, \Delta T_{m3}$ = corresponding log-mean temperature driving forces
T_c = final thermal center temperature
T_{fm} = mean freezing point, assumed 1.5 K below initial freezing point
T_o = mean final temperature
V = volume of food item

Table 7 Geometric Constants

Shape	G_1	G_2	G_3
Infinite slab	1	0	0
Infinite cylinder	2	0	0
Sphere	3	0	0
Squat cylinder	1	2	0
Short cylinder	2	0	1
Infinite rod	1	1	0
Rectangular brick	1	1	1
Two-dimensional irregular shape	1	1	0
Three-dimensional irregular shape	1	1	1

Source: Cleland et al. (1987a)

Hossain et al. (1992b) also presented a semi-analytically derived expression for the equivalent heat transfer dimensionality of two-dimensional, irregularly shaped food items. An equivalent "pseudo-elliptical" infinite cylinder was used to replace the actual two-dimensional, irregular shape in the calculations. A pseudo-ellipse is a shape that depends on the Biot number. As the Biot number approaches infinity, the shape closely resembles an ellipse. As the Biot number approaches zero, the pseudo-elliptical infinite cylinder approaches an infinite rectangular rod. Hossain et al. (1992b) stated that for practical Biot numbers, the pseudo-ellipse is very similar to a true ellipse. This model pseudo-elliptical infinite cylinder has the same volume per unit length and characteristic dimension as the actual food item. The resulting expression for E is given as follows:

$$E = 1 + \frac{1 + \dfrac{2}{\mathrm{Bi}}}{\beta^2 + \dfrac{2\beta}{\mathrm{Bi}}} \tag{65}$$

In Equation (65), the Biot number is based on the shortest distance from the thermal center to the surface of the food item; not twice that distance. Using this expression for E, the freezing time of two-dimensional, irregularly shaped food items θ_{shape} can be calculated via Equation (56).

Hossain et al. (1992c) extended this analysis to the prediction of freezing times of three-dimensional, irregularly shaped food items. In this work, the irregularly shaped food item was replaced with a model ellipsoid shape having the same volume, characteristic dimension and smallest cross sectional area orthogonal to the characteristic dimension, as the actual food item. An expression was presented for E of a pseudo-ellipsoid as follows:

$$E = 1 + \frac{1 + \dfrac{2}{\mathrm{Bi}}}{\beta_1^2 + \dfrac{2\beta_1}{\mathrm{Bi}}} + \frac{1 + \dfrac{2}{\mathrm{Bi}}}{\beta_2^2 + \dfrac{2\beta_2}{\mathrm{Bi}}} \tag{66}$$

In Equation (66), the Biot number is based upon the shortest distance from the thermal center to the surface of the food item; not twice that distance. With this expression for E, the freezing time of three-dimensional, irregularly shaped food items θ_{shape} may be calculated using Equation (56).

The method of Lin et al. (1996a, 1996b), which was discussed in the Cooling Times of Foods and Beverages section, Equations (16) through (23) and Table 4, may also be used to determine equivalent heat transfer dimensionality for freezing time calculations. It applies to all the geometric shapes given in Table 4.

Table 9 summarizes the numerous methods that have been discussed for determining the equivalent heat transfer dimensionality of various geometries. These methods can be used in conjunction with Equation (14) to calculate cooling times or in conjunction with Equation (56) to calculate freezing times.

Mean Conducting Path. Pham's freezing time formulas, given in Equations (51) and (52), require knowledge of the Biot number. To calculate the Biot number of a food, its characteristic dimension must be known. Because it is difficult to determine the characteristic dimension of an irregularly shaped food, Pham (1985) introduced the concept of the **mean conducting path**. The mean conducting path is the mean heat transfer length from the surface of the food item to its thermal center, or $D_m/2$. Thus, the Biot number becomes:

$$\mathrm{Bi} = \frac{hD_m}{k} \tag{67}$$

where D_m is twice the mean conducting path.

For rectangular blocks of food, Pham (1985) found that the mean conducting path was proportional to the geometric mean of the block's two shorter dimensions. Based on this result, Pham (1985) presented an equation to calculate the Biot number for rectangular blocks of food:

$$\frac{\mathrm{Bi}}{\mathrm{Bi}_o} = 1 + \left\{[1.5\sqrt{\beta_1} - 1]^{-4} + \left[\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)\left(1 + \frac{4}{\mathrm{Bi}_o}\right)\right]^{-4}\right\}^{-0.25} \tag{68}$$

where Bi_o is the Biot number based on the shortest dimension of the block D_1, or $\mathrm{Bi}_o = hD_1/k$. The Biot number calculated with Equation (68) can then be substituted into a freezing time estimation method to calculate the freezing time for rectangular blocks.

Table 8 Expressions for Equivalent Heat Transfer Dimensionality

Shape	**Expressions for Equivalent Heat Transfer Dimensionality, *E***
Infinite rectangular rod ($2L$ by $2\beta_1 L$)	$E = \left(1 + \frac{2}{\mathrm{Bi}}\right)\left\{\left(1 + \frac{2}{\mathrm{Bi}}\right) - 4\sum_{n=1}^{\infty}\left[\frac{(\sin z_n)}{z_n^3\left(1 + \frac{\sin^2 z_a}{\mathrm{Bi}}\right)\left(\frac{z_n}{\mathrm{Bi}}\sinh(z_n\beta_1) + \cosh(z_n\beta_1)\right)}\right]\right\}^{-1}$ where z_n are roots of $\mathrm{Bi} = z_n \tan(z_n)$ and $\mathrm{Bi} = hL/k$ where L is the shortest distance from the center of the rectangular rod to the surface
Finite cylinder, height exceeds diameter (radius L and height $2\beta_1 L$)	$E = \left(2 + \frac{4}{\mathrm{Bi}}\right)\left\{\left(1 + \frac{2}{\mathrm{Bi}}\right) - 8\sum_{n=1}^{\infty}\left[y_n^3 J_1(y_n)\left(1 + \frac{y_n^2}{\mathrm{Bi}^2}\right)\left(\cosh(\beta_1 y_n) + \frac{y_n}{\mathrm{Bi}}\sinh(\beta_1 y_n)\right)\right]^{-1}\right\}^{-1}$ where y_n are roots of $y_n J_1(y_n) - \mathrm{Bi}\, J_0(y_n) = 0$; J_0 and J_1 are Bessel functions of the first kind, order zero and one, respectively; and $\mathrm{Bi} = hL/k$ where L is the radius of the cylinder.
Finite cylinder, diameter exceeds height (radius $\beta_1 L$ and height $2L$)	$E = \left(1 + \frac{2}{\mathrm{Bi}}\right)\left\{\left(1 + \frac{2}{\mathrm{Bi}}\right) - 4\sum_{n=1}^{\infty}\frac{\sin z_n}{z_n^2(z_n + \cos z_n \sin z_n)\left(I_0(z_n\beta_1) + \frac{z_n}{\mathrm{Bi}} I_1(z_n\beta_1)\right)}\right\}^{-1}$ where z_n are roots of $\mathrm{Bi} = z_n \tan(z_n)$; I_0 and I_1 are Bessel function of the second kind, order zero and one, respectively; and $\mathrm{Bi} = hL/k$ where L is the radius of the cylinder.
Rectangular Brick ($2L$ by $2\beta_1 L$ by $2\beta_2 L$)	$E = \left(1 + \frac{2}{\mathrm{Bi}}\right)\left\{\left(1 + \frac{2}{\mathrm{Bi}}\right) - 4\sum_{n=1}^{\infty}\left[\frac{\sin z_n}{z_n^3\left(1 + \frac{\sin^2 z_n}{\mathrm{Bi}}\right)\left[\frac{z_n}{\mathrm{Bi}}\sinh(z_n\beta_1) + \cosh(z_n\beta_1)\right]}\right]\right.$ $- 8\beta_2^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\sin z_n \sin z_m\left[\left(\cosh(z_{nm}) + \frac{z_{nm}}{\mathrm{Bi}\beta_2}\sinh(z_{nm})\right)\right.\right.$ $\left.\left.\left. z_n z_m z_{nm}^2\left(1 + \frac{1}{\mathrm{Bi}}\sin^2 z_n\right)\left(1 + \frac{1}{\mathrm{Bi}\beta_1}\sin^2 z_m\right)\right]^{-1}\right]\right\}^{-1}$ where z_n are roots of $\mathrm{Bi} = z_n \tan(z_n)$, z_m are the roots of $\mathrm{Bi}\beta_1 = z_m \tan(z_m)$, $\mathrm{Bi} = hL/k$ where L is the shortest distance from the thermal center of the rectangular brick to the surface, and z_{nm} is given as: $z_{nm}^2 = z_n^2\beta_2^2 + z_m^2\left(\frac{\beta_2}{\beta_1}\right)^2$

Source: Hossain et al. (1992a)

Table 9 Summary of Methods for Determining Equivalent Heat Transfer Dimensionality

Slab		Cleland et al. (1987a, 1987b) Equations (61) – (64)		Lin et al. (1996a, 1996b) Equations (16) – (23)
Infinite Cylinder		Cleland et al. (1987a, 1987b) Equations (61) – (64)		Lin et al. (1996a, 1996b) Equations (16) – (23)
Sphere		Cleland et al. (1987a, 1987b) Equations (61) – (64)		Lin et al. (1996a, 1996b) Equations (16) – (23)
Squat cylinder		Cleland et al. (1987a, 1987b) Equations (61) – (64)	Hossain et al. (1992a) Table 8	Lin et al. (1996a, 1996b) Equations (16) – (23)
Short cylinder	Cleland and Earle (1982) Equations (59) and (60)	Cleland et al. (1987a, 1987b) Equations (61) – (64)	Hossain et al. (1992a) Table 8	Lin et al. (1996a, 1996b) Equations (16) – (23)
Infinite rod		Cleland et al. (1987a, 1987b) Equations (61) – (64)	Hossain et al. (1992a) Table 8	Lin et al. (1996a, 1996b) Equations (16) – (23)
Rectangular brick	Cleland and Earle (1982) Equations (57), (58), and (59)	Cleland et al. (1987a, 1987b) Equations (61) – (64)	Hossain et al. (1992a) Table 8	Lin et al. (1996a, 1996b) Equations (16) – (23)
2-D Irregular shape (infinite ellipse)		Cleland et al. (1987a, 1987b) Equations (61) – (64)	Hossain et al. (1992b) Equation (65)	Lin et al. (1996a, 1996b) Equations (16) – (23)
3-D Irregular shape (ellipsoid)		Cleland et al. (1987a, 1987b) Equations (61) – (64)	Hossain et al. (1992b) Equation (66)	Lin et al. (1996a, 1996b) Equations (16) – (23)

Pham (1985) noted that for squat shaped food items the mean conducting path $D_m/2$ could be reasonably estimated as the arithmetic mean of the longest and shortest distances from the surface of the food item to its thermal center.

Equivalent Sphere Diameter. Ilicali and Hocalar (1990) and Ilicali and Engez (1990) introduced the **equivalent sphere diameter** concept to calculate the freezing time of irregularly shaped food items. In this method, a sphere diameter is calculated that is based on the volume and the volume to surface area ratio of the irregularly shaped food. This equivalent sphere is then used to calculate the freezing time of the food item.

Considering an irregularly shaped food item where the shortest and longest distances from the surface to the thermal center were designated as D_1 and D_2, respectively, Ilicali and Hocalar (1990) and Ilicali and Engez (1990) defined the volume-surface diameter D_{vs} as the diameter of a sphere having the same volume to surface area ratio as the irregular shape:

$$D_{vs} = 6V/A_s \tag{69}$$

where V is the volume of the irregular shape and A_s is the surface area of the irregular shape. In addition, the volume diameter D_v is defined as the diameter of a sphere having the same volume as the irregular shape:

$$D_v = (6V/\pi)^{1/3} \tag{70}$$

Because a sphere is the solid geometry which has minimum surface area per unit volume, the equivalent sphere diameter $D_{eq,s}$, must be greater than D_{vs} and smaller than D_v. In addition, the contribution of the volume diameter D_v has to decrease as the ratio of the longest to the shortest dimensions D_2/D_1 increases, because the object will be essentially two dimensional if $D_2/D_1 >> 1$. Therefore, the equivalent sphere diameter $D_{eq,s}$ is defined as follows:

$$D_{eq,s} = \frac{1}{\beta_2 + 1} D_v + \frac{\beta_2}{\beta_2 + 1} D_{vs} \tag{71}$$

Thus, the prediction of the freezing time of the irregularly shaped food item is reduced to predicting the freezing time of a spherical food item with diameter $D_{eq,s}$. Any of the previously discussed freezing time methods for spheres may then be used to calculate this freezing time.

Algorithms for Freezing Time Estimation

The following suggested algorithm for estimating the freezing time of foods and beverages is based on the modified Plank equation presented by Cleland and Earle (1977, 1979a, 1979b). This algorithm is applicable to simple food geometries, including infinite slabs, infinite cylinders, spheres, and three-dimensional rectangular bricks.

1. Determine thermal properties of food item (see Chapter 8).
2. Determine surface heat transfer coefficient for the freezing process (see Chapter 8).
3. Determine characteristic dimension D and dimensional ratios β_1 and β_2 using Equations (17) and (18).
4. Calculate Biot number, Plank number, and Stefan number using Equations (29), (30), and (31), respectively.
5. Determine geometric parameters P and R given in Table 5.
6. Calculate freezing time using Equation (32) or Equation (33) depending on the final temperature of the frozen food.

The following algorithm for estimating the freezing time of foods and beverages is based on the method of equivalent heat transfer dimensionality. This algorithm is applicable to many food geometries, including infinite rectangular rods, finite cylinders, three-dimensional rectangular bricks, and two- and three-dimensional irregular shapes.

1. Determine thermal properties of the food item (see Chapter 8).
2. Determine surface heat transfer coefficient for the freezing process (see Chapter 8).
3. Determine characteristic dimension D and dimensional ratios β_1 and β_2 using Equations (17) and (18).
4. Calculate Biot number, Plank number, and Stefan number using Equations (29), (30), and (31), respectively.
5. Calculate freezing time of an infinite slab using a suitable method. Suitable methods include:
   (a) Equation (32) or (33) in conjunction with the geometric parameters P and R given in Table 5.
   (b) Equation (37) or (38) in conjunction with Equations (34), (35), and (36).
6. Calculate equivalent heat transfer dimensionality for the food item. Refer to Table 9 to determine which equivalent heat transfer dimensionality method is applicable to the particular food geometry.
7. Calculate the freezing time of the food item using Equation (56).

SAMPLE PROBLEMS FOR ESTIMATING FREEZING TIME

Example 3. A rectangular brick shaped package of beef (lean sirloin) with dimension 0.04 m by 0.12 m by 0.16 m is to be frozen in an air blast freezer. The initial temperature of the beef is 10°C and the freezer air temperature is −30°C. It is estimated that the surface heat transfer coefficient is 40 W/(m²·K). Calculate the time required for the thermal center of the beef to reach a temperature of −10°C.

Solution: Use the algorithm based on the modified Plank equation by Cleland and Earle (1977, 1979a, 1979b).

Step 1: Determine the thermal properties of lean sirloin.

Using the methods described in Chapter 8, the thermal properties can be calculated as follows:

Property	At −40°C (Fully Frozen)	At −10°C (Final Temp.)	At −1.7°C (Initial Freezing Point)	At 10°C (Initial Temp.)
Density, kg/m³	$\rho_s = 1018$	$\rho_s = 1018$	$\rho_l = 1075$	$\rho_l = 1075$
Enthalpy, kJ/kg	—	$H_s = 83.4$	$H_l = 274.2$	—
Specific heat, kJ/(kg·K)	$c_s = 2.11$	—	—	$c_l = 3.52$
Thermal conductivity, W/(m·K)	$k_s = 1.66$	—	—	—

Volumetric enthalpy difference between the initial freezing point and −10°C:

$$\Delta H_{10} = \rho_l H_l - \rho_s H_s$$

$$\Delta H_{10} = (1075)(274.2) - (1018)(83.4) = 210 \times 10^3 \text{ kJ/m}^3$$

Volumetric specific heats:

$$C_s = \rho_s c_s = (1018)(2.11) = 2148 \text{ kJ/(m}^3\cdot\text{K)}$$

$$C_l = \rho_l c_l = (1075)(3.52) = 3784 \text{ kJ/(m}^3\cdot\text{K)}$$

Step 2: Determine the surface heat transfer coefficient.

The surface heat transfer coefficient is estimated to be 40 W/(m²·K).

Step 3: Determine the characteristic dimension D and the dimensional ratios β_1 and β_2.

For freezing time problems, the characteristic dimension D is twice the shortest distance from the thermal center of the food item to its surface. For this example, $D = 0.04$ m.

Using Equations (17) and (18), the dimensional ratios then become:

$$\beta_1 = 0.12/0.04 = 3$$

$$\beta_2 = 0.16/0.04 = 4$$

Step 4: Using Equations (29) through (31), calculate the Biot number, the Plank number, and the Stefan number.

$$\text{Bi} = \frac{hD}{k_s} = \frac{(40.0)(0.04)}{1.66} = 0.964$$

$$\text{Pk} = \frac{C_l(T_i - T_f)}{\Delta H_{10}} = \frac{(3784)[10-(-1.7)]}{210\times10^3} = 0.211$$

$$\text{Ste} = \frac{C_s(T_f - T_m)}{\Delta H_{10}} = \frac{(2148)[-1.7-(-30)]}{210\times10^3} = 0.289$$

Step 5: Determine the geometric parameters P and R for the rectangular brick.

Determine P from Table 5.

$$P_1 = \frac{(3)(4)}{2[(3)(4)+3+4]} = 0.316$$

$$P_2 = 0.316\left\{1.026 + (0.5808)(0.211) + 0.289\left[(0.2296)(0.211) + \frac{0.0182}{0.964} + 0.1050\right]\right\}$$

$$P_2 = 0.379$$

$$P = 0.379 + 0.316\{0.1136 + 0.289[(5.766)(0.316) - 1.242]\}$$

$$P = 0.468$$

Determine R from Table 5.

$$\frac{1}{Q} = 4[(3-4)(3-1)+(4-1)^2]^{1/2} = 10.6$$

$$r = \frac{1}{3}\{3+4+1+[(3-4)(3-1)+(4-1)^2]^{1/2}\} = 3.55$$

$$s = \frac{1}{3}\{3+4+1-[(3-4)(3-1)+(4-1)^2]^{1/2}\} = 1.78$$

$$R_1 = \frac{1}{(10.6)(2)}\left\{(3.55-1)(3-3.55)(4-3.55)\ln\left[\frac{3.55}{3.55-1}\right] - (1.78-1)(3-1.78)(4-1.78)\ln\left[\frac{1.78}{1.78-1}\right]\right\} + \frac{1}{72}[(2)(3)+(2)(4)-1] = 0.0885$$

$$R_2 = 0.0885\{1.202 + 0.289[(3.410)(0.211) + 0.7336]\} = 0.144$$

$$R = 0.144 + 0.0885\{0.7344 + 0.289[(49.89)(0.0885) - 2.900]\} = 0.248$$

Step 6: Calculate the freezing time of the beef.

Because the final temperature at the thermal center of the beef is given to be −10°C, use Equation (32) to calculate the freezing time:

$$\theta = \frac{2.10\times10^8}{-1.7-(-30)}\left[\frac{(0.468)(0.04)}{40.0} + \frac{(0.248)(0.04)^2}{1.66}\right] = 5250 \text{ s} = 1.46 \text{ h}$$

Example 4. Orange juice in a cylindrical container, 0.30 m diameter by 0.45 m tall, is to be frozen in an air blast freezer. The initial temperature of the juice is 5°C and the freezer air temperature is −35°C. It is estimated that the surface heat transfer coefficient is 30 W/(m²·K). Calculate the time required for the thermal center of the juice to reach a temperature of −18°C.

Solution: Use the algorithm based on the method of equivalent heat transfer dimensionality.

Step 1: Determine the thermal properties of orange juice.

Using the methods described in Chapter 8, the thermal properties of orange juice are calculated as follows:

Property	At −40°C (Fully Frozen)	At −18°C (Final Temp.)	At 5°C (Initial Temp.)
Density, kg/m³	ρ_s = 970	ρ_s = 970	ρ_l = 1038
Enthalpy, kJ/kg	—	H_s = 40.8	H_l = 381.5
Specific Heat, kJ/(kg·K)	c_s = 1.76	—	c_l = 3.89
Thermal Cond., W/(m·K)	k_s = 2.19	—	—
Initial Freezing Temperature: T_f = −0.4°C			

Volumetric enthalpy difference between T_i = 5°C, and −18°C:

$$\Delta H_{18} = \rho_l H_l - \rho_s H_s$$

$$\Delta H_{18} = (1038)(381.5) - (970)(40.8) = 356\times10^3 \text{ kJ/m}^3$$

Volumetric specific heats:

$$C_s = \rho_s c_s = (970)(1.76) = 1707 \text{ kJ/(m}^3\cdot\text{K)}$$

$$C_l = \rho_l c_l = (1038)(3.89) = 4038 \text{ kJ/(m}^3\cdot\text{K)}$$

Step 2: Determine the surface heat transfer coefficient.

The surface heat transfer coefficient is estimated to be 30 W/(m²·K).

Step 3: Determine the characteristic dimension D and the dimensional ratios β_1 and β_2.

For freezing time problems, the characteristic dimension is twice the shortest distance from the thermal center of the food item to its surface. For the cylindrical sample of orange juice, the characteristic dimension is equal to the diameter of the cylinder:

$$D = 0.30 \text{ m}$$

Using Equations (17) and (18), the dimensional ratios then become:

$$\beta_1 = \beta_2 = \frac{0.45 \text{ m}}{0.30 \text{ m}} = 1.5$$

Step 4: Using Equations (29) through (31), calculate the Biot number, the Plank number, and the Stefan number.

$$\text{Bi} = \frac{hD}{k_s} = \frac{(30.0)(0.30)}{2.19} = 4.11$$

$$\text{Pk} = \frac{C_l(T_i - T_f)}{\Delta H_{18}} = \frac{(4038)[5-(-0.4)]}{356\times10^3} = 0.0613$$

$$\text{Ste} = \frac{C_s(T_f - T_m)}{\Delta H_{18}} = \frac{(1707)[-0.4-(-35)]}{356\times10^3} = 0.166$$

Step 5: Calculate the freezing time of an infinite slab.

Use the method of Hung and Thompson (1983). First, find the weighted average temperature difference given by Equation (34).

$$\Delta T = [-0.4-(-35)] + \frac{[5-(-0.4)]^2(4038/2) - [-0.4-(-18)]^2(1707/2)}{356\times10^3} = 34.0 \text{ K}$$

Determine the parameter U:

$$U = \frac{34.0}{-0.4-(-35)} = 0.983$$

Determine the geometric parameters, P and R, for an infinite slab using Equations (35) and (36):

$$P = 0.7306 - (1.083)(0.0613) + (0.166)\left[(15.40)(0.983) - 15.43 + \frac{(0.01329)(0.166)}{4.11}\right] = 0.616$$

$$R = 0.2079 - (0.2656)(0.983)(0.166) = 0.165$$

Determine the freezing time of the slab using Equation (37):

$$\theta = \frac{3.56\times10^8}{34.0}\left[\frac{(0.616)(0.30)}{30.0} + \frac{(0.165)(0.30)^2}{2.19}\right] = 135000 \text{ s} = 37.5 \text{ h}$$

Step 6: Calculate the equivalent heat transfer dimensionality for a finite cylinder.

Use the method presented by Cleland et al. (1987a, 1987b), Equations (61) through (64), to calculate the equivalent heat transfer dimensionality. From Table 7, the geometric constants for a cylinder are:

$$G_1 = 2 \qquad G_2 = 0 \qquad G_3 = 1$$

Calculate E_2:

$$E_2 = \frac{2.32}{\beta_2^{1.77}} = \frac{2.32}{1.5^{1.77}} = 1.132$$

$$X(1.132) = \frac{1.132}{4.11^{1.34} + 1.132} = 0.146$$

$$E_2 = \frac{0.146}{1.5} + (1 - 0.146)\frac{0.73}{1.5^{2.50}} = 0.324$$

Thus, the equivalent heat transfer dimensionality E becomes:

$$E = G_1 + G_2E_1 + G_3E_2$$

$$E = 2 + (0)(E_1) + (1)(0.324) = 2.324$$

Step 7: Calculate freezing time of the orange juice using Equation (56):

$$\theta_{shape} = \theta_{slab}/E = 135000/2.324 = 58100 \text{ s} = 16.1 \text{ h}$$

NOMENCLATURE

A_1 = cross sectional area in Equation (9), m^2
A_2 = cross sectional area in Equation (9), m^2
A_s = surface area of food item, m^2
B_1 = parameter in Equation (8)
B_2 = parameter in Equation (8)
Bi = Biot number
Bi_1 = Biot number for precooling = $(Bi_l + Bi_s)/2$
Bi_2 = Biot number for phase change = Bi_s
Bi_3 = Biot number for subcooling = Bi_s
Bi_c = Biot number evaluated at $k_c = hD/k_c$
Bi_l = Biot number for unfrozen food = hD/k_l
Bi_o = Biot number based on shortest dimension = hD_1/k
Bi_s = Biot number for fully frozen food = hD/k_s
c = specific heat of food item, J/(kg·K)
C_l = volumetric specific heat of unfrozen food, J/(m^3·K)
C_s = volumetric specific heat of fully frozen food, J/(m^3·K)
D = slab thickness or cylinder/sphere diameter, m
D_1 = shortest dimension, m
D_2 = longest dimension, m
$D_{eq,s}$ = equivalent sphere diameter, m
D_m = twice the mean conducting path, m
D_v = volume diameter, m
D_{vs} = volume-surface diameter, m
E = equivalent heat transfer dimensionality
E_o = equivalent heat transfer dimensionality at Bi = 0
E_1 = parameter given by Equation (62)
E_2 = parameter given by Equation (63)
E_∞ = equivalent heat transfer dimensionality at Bi $\to \infty$
f = cooling time parameter
f_1 = cooling time parameter for precooling
f_3 = cooling time parameter for subcooling
f_{comp} = cooling parameter for a composite shape
G = geometry index
G_1 = geometric constant in Equation (61)
G_2 = geometric constant in Equation (61)
G_3 = geometric constant in Equation (61)
h = heat transfer coefficient, W/(m^2·K)
$I_o(x)$ = Bessel function of the second kind, order zero
$I_1(x)$ = Bessel function of the second kind, order one
j = cooling time parameter
j_1 = cooling time parameter for precooling
j_3 = cooling time parameter for subcooling
j_c = cooling time parameter applicable to thermal center
j_{comp} = cooling time parameter for a composite shape
j_m = cooling time parameter applicable to mass average
$J_o(x)$ = Bessel function of the first kind, order zero
$J_1(x)$ = Bessel function of the first kind, order one
k = thermal conductivity of food item, W/(m·K)
k_c = thermal conductivity of food evaluated at $(T_f + T_m)/2$, W/(m·K)
k_l = thermal conductivity of unfrozen food, W/(m·K)
k_s = thermal conductivity of fully frozen food, W/(m·K)
L = half thickness of slab or radius of cylinder/sphere, m
L_f = volumetric latent heat of fusion, J/m^3
L_∞ = lag factor parameter given by Equation (25)
m = inverse of Biot number
M_1^2 = characteristic value of Smith et al. (1968)
N = number of dimensions
p_1 = geometric parameter from Lin et al. (1996b)
p_2 = geometric parameter from Lin et al. (1996b)
p_3 = geometric parameter from Lin et al. (1996b)
P = Plank's geometry factor
P' = geometric factor for rectangular bricks calculated using method in Table 5
P_1 = intermediate value of Plank's geometric factor
P_2 = intermediate value of Plank's geometric factor
Pk = Plank number = $C_l(T_i - T_f)/\Delta H$
Q = parameter given in Table 5
Q_1 = volumetric heat of precooling, J/m^3
Q_2 = volumetric heat of phase change, J/m^3
Q_3 = volumetric heat of subcooling, J/m^3
r = parameter given in Table 5
R = Plank's geometry factor
R' = geometric factor for rectangular bricks calculated using method in Table 5
R_1 = intermediate value of Plank's geometric factor
R_2 = intermediate value of Plank's geometric factor
s = parameter given in Table 5
Ste = Stefan number = $C_s(T_f - T_m)/\Delta H$
T = product temperature, °C
T_c = final center temperature of food item, °C
T_f = initial freezing temperature of food item, °C
T_{fm} = mean freezing temperature, °C
T_i = initial temperature of food item, °C
T_m = cooling or freezing medium temperature, °C
T_o = mean final temperature, °C
T_{ref} = reference temperature for freezing time correction factor, °C
u = parameter given in Table 1
U = parameter in Equations (35) and (36) = $\Delta T/(T_f - T_m)$
v = parameter given in Table 2
V = volume of food item, m^3
w = parameter given in Table 3
W_1 = parameter given by Equation (58)
W_2 = parameter given by Equation (59)
x = coordinate direction
$X(x)$ = function given by Equation (64)
X_b = parameter in Equation (13)
X_g = parameter in Equations (12) and (13)
y = coordinate direction
Y = fractional unaccomplished temperature difference
y_n = roots of transcendental equation; $y_nJ_1(y_n) - BiJ_o(y_n) = 0$
z = coordinate direction
z_m = roots of transcendental equation; $Bi\beta_1 = z_m\tan(z_m)$
z_n = roots of transcendental equation; $Bi = z_n\tan(z_n)$
z_{nm} = parameter given in Table 8
α = thermal diffusivity of food, m^2/s

β_1 = ratio of second shortest dimension to shortest dimension, Equation (17)
β_2 = ratio of longest dimension to shortest dimension, Equation (18)
γ_1 = geometric parameter from Lin et al. (1996b)
γ_2 = geometric parameter from Lin et al. (1996b)
θ = cooling or freezing time, s
θ_1 = precooling time, s
θ_2 = phase change time, s
θ_3 = tempering time, s
θ_{shape} = freezing time of an irregular shaped food item, s
θ_{slab} = freezing time of an infinite slab shaped food item, s
ΔH = volumetric enthalpy difference, J/m^3
ΔH_1 = volumetric enthalpy difference = $C_l(T_i - T_{fm})$, J/m^3
ΔH_2 = volumetric enthalpy difference = $L_f + C_s(T_{fm} - T_c)$, J/m^3
ΔH_{10} = volumetric enthalpy difference between the initial freezing temperature T_f and −10°C, J/m^3
ΔH_{18} = volumetric enthalpy difference between initial temperature T_i and −18°C, J/m^3
ΔT = weighted average temperature difference given by Equation (34), °C
ΔT_1 = temperature difference = $(T_i + T_{fm})/2 - T_m$, °C
ΔT_2 = temperature difference = $T_{fm} - T_m$, °C
ΔT_{m1} = temperature difference for precooling, °C
ΔT_{m2} = temperature difference for phase change, °C
ΔT_{m3} = temperature difference for subcooling, °C
λ = geometric parameter from Lin et al. (1996b)
μ = parameter given by Equation (27)
ρ = density of food item, kg/m^3
ω = first root of Equation (15)

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